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Inertial Frames and Tidal Forces Along the Symmetry Axis of the Kerr Spacetime

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The gravitational field along the symmetry axis of the Kerr spacetime is examined. The equations of parallel transport along this axis are solved for the timelike geodesics case, and the corresponding tidal tensor is constructed.

1. INTRODUCTION

As in every axially symmetric case in physics, the symmetry axis of the Kerr spacetime should provide the ground for a simplified analysis of the gravitational field corresponding to the celebrated Kerr metric [1]. This has proved actually to be the case as shown, for example, by the fact that Carter [2] first obtained a maximal analytic extension of the submanifold of the Kerr spacetime consisting of events occurring on its symmetry axis before he accomplished the same thing for the spacetime manifold itself [3].

In this paper we show that the simplification that is obtained on the axis of symmetry allows for an exact and intuitively clear treatment of effects which are due to the nonstatic character of the Kerr field such as the rotation induced on inertial frames falling along the axis. The behavior of inertial frames is analyzed in Section 3, where other classes of frames, such as the locally nonrotating ones, are also examined. Our analysis is based on the Kerr–Schild [4] form of Kerr's metric which is presented in Section 2, along with notation. In the fourth and final section the tidal tensor is determined, i.e., the tensor on the basis of which one can study the physics of the field of gravity inside an inertial frame. The construction of

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inertial frames along the symmetry axis and the calculation of the tidal tensor presented in this paper completes the important work of Marck [5] who has solved the same problem for the case of geodesics lying off the axis of the Kerr spacetime.

2. COORDINATE SYSTEMS AND THE METRIC

In the Boyer–Lindquist (BL) coordinate system (t, r, θ, φ) the Kerr metric reads [6]

$$\begin{aligned} ds^2 = & -(1-2Mr/\Sigma)\, dt^2 - 2(2Mr/\Sigma)\, a \sin^2\theta \, dt d\varphi \\ & + (\Sigma/\Delta)\, dr^2 + \Sigma\, d\theta^2 + (A/\Sigma)\sin^2\theta\, d\varphi^2 \end{aligned} \tag{1}$$

where

$$\begin{aligned} \Sigma &:= r^2 + a^2\cos^2\theta \\ \Delta &:= r^2 + a^2 - 2Mr \\ A &:= (r^2+a^2)^2 - \Delta a^2 \sin^2\theta \end{aligned} \tag{2}$$

and M, a stand for the mass and specific angular momentum, respectively.

The axis of symmetry consists of the points where $\sin\theta = 0$. At such points, however, the angular coordinate φ is undefined. Thus, in order to study the geometry of the Kerr spacetime along its symmetry axis and the neighborhood of the latter, one has to shift from the BL coordinate system to one that is well-defined at the points corresponding to $\theta = 0, \pi$.

A coordinate system which is regular along the axis is that of Kerr and Schild (KS). In the KS coordinates (T, x, y, z) the Kerr metric takes the form [4]

$$\begin{aligned} ds^2 = & -dT^2 + dx^2 + dy^2 + dz^2 + \frac{2Mr^3}{r^4 + a^2 z^2} \\ & \times \left\{ -dT + \frac{1}{r^2+a^2}\left[r(x\,dx + y\,dy) + a(x\,dy - y\,dx)\right]\frac{z}{r} + dz \right\}^2 \end{aligned} \tag{3}$$

where the function $r(x, y, z)$ is implicitly defined by the equation

$$\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1 \tag{4}$$

The relation between the BL and KS systems of coordinates is expressed by the following set of equations

$$
\begin{aligned}
dT &= dt - (2Mr/\Delta)\, dr \\
d\psi &= d\varphi - \frac{2Mar}{(r^2+a^2)\,\Delta}\, dr \\
x &= (r^2+a^2)^{1/2} \sin\theta \cos\psi \\
y &= (r^2+a^2)^{1/2} \sin\theta \cos\psi \\
z &= r\cos\theta
\end{aligned}
\tag{5}
$$

These equations show that in the KS coordinates the symmetry axis of the Kerr spacetime is defined by the condition

$$x = y = 0, \qquad |z| = r \tag{6}$$

Thus, on the axis itself, the metric (3) takes the simple form

$$ds^2 \overset{*}{=} -dT^2 + dx^2 + dy^2 + dz^2 + 2Mr/(z^2+a^2)(-dT+dz)^2 \tag{7}$$

where the asterisk over the equals sign denotes the fact that the given equation holds along the axis of symmetry only.

It should be noted that (4) allows r to take negative values. This corresponds to the fact that the Kerr metric can be analytically extended to regions where $r<0$, as shown [2, 3]. Such regions can be mapped in a chart with coordinates (T', x', y', z') in which the metric has the same form as that given by (3) except for a change of the sign in front of the braces (curly brackets). The bottom side of the disc $x^2+y^2 \leqslant a^2, z=0$ is then identified with the top side of the corresponding disc in the primed coordinates, and vice-versa. Details of this construction can be found [7].

Equation (4) implies that the vector l_c, where

$$l_c := \left(-1, \frac{rx-ay}{r^2+a^2}, \frac{ry+ax}{r^2+a^2}, \frac{z}{r}\right) \tag{8}$$

and $c = 0, 1, 2, 3$ corresponding to T, x, y, z is a null vector with respect to the Minkowski metric $\eta_{ab} := \mathrm{diag}(-1, 1, 1, 1)$, i.e.

$$\eta^{ab} l_a l_b = 0 \tag{9}$$

Furthermore, it follows from (3) and (8) that the Kerr metric g_{ab} can be written in the form

$$g_{ab} = \eta_{ab} + 2ML^2 l_a l_b \tag{10}$$

where

$$L^2 := r^3/(r^4 + a^2 z^2) \tag{11}$$

It then follows from (9) that

$$g^{ab} := \eta^{ab} - 2ML^2 l^a l^b \tag{12}$$

is the inverse of g_{ab} where

$$l^a := \eta^{ab} l_b \tag{13}$$

Also

$$l^a = g^{ab} l_b \tag{14}$$

Thus, l_a is a null vector with respect to the Kerr metric, too.

Equations (6)–(14) are used in Appendices A and B for the calculation of the connection coefficients and the curvature tensor of g_{ab} along the symmetry axis of the Kerr spacetime. These quantities are needed in the following sections.

3. STATIONARY AND INERTIAL FRAMES

Let us consider a particle confined on the symmetry axis of the Kerr spacetime at fixed z. According to (7), such a particle's 4-velocity $e_{\hat{0}}{}^a$ and the vectors $e_{\hat{i}}{}^a$, $i = 1, 2, 3$, where

$$\begin{aligned}
e_{\hat{0}}{}^a &:\overset{*}{=} [(z^2 + a^2)/\Delta]^{1/2}\, \delta_0{}^a \\
e_{\hat{1}}{}^a &:\overset{*}{=} \delta_1{}^a \\
e_{\hat{2}}{}^a &:\overset{*}{=} \delta_2{}^a \\
e_{\hat{3}}{}^a &:\overset{*}{=} [\Delta/(z^2 + a^2)]^{1/2}\, [\delta_3{}^a - (2Mz/\Delta)\, \delta_0{}^a]
\end{aligned} \tag{15}$$

form an orthonormal tetrad. Since its spatial legs $e_{\hat{i}}{}^a$ have the direction of the coordinate axes x^i, the frame (15) is not rotating relative to the fixed stars.

Using the covariant vectors $\{e_{\hat{a}b}\}$, where

$$e_{\hat{a}b} := g_{bc} e_{\hat{a}}{}^c \tag{16}$$

we can define an orthonormal tetrad of 1-forms dual to the tetrad $\{e_{\hat{a}}{}^b\}$ given above. This is accomplished by letting

$$w^{\hat{a}}{}_c := \eta^{\hat{a}\hat{b}} e_{\hat{b}c} \tag{17}$$

and then (5)–(7) and (15), (16) give

$$
\begin{aligned}
w^{\hat{0}}{}_a\, dx^a &= \left(\frac{\Delta}{z^2+a^2}\right)^{1/2}\left(dT + \frac{2Mz}{\Delta}\, dz\right) = \left(\frac{\Delta}{z^2+a^2}\right)^{1/2} dt \\
w^{\hat{1}}{}_a\, dx^a &= dx \\
w^{\hat{2}}{}_a\, dx^a &= dy \\
w^{\hat{3}}{}_a\, dx^a &= [(z^2+a^2)/\Delta]^{1/2}\, dz
\end{aligned} \tag{18}
$$

and

$$
\begin{aligned}
ds^2 &\stackrel{*}{=} \eta_{\hat{a}\hat{b}} w^{\hat{a}} w^{\hat{b}} \\
&= -\left(1 - \frac{2Mr}{z^2+a^2}\right) dt^2 + dx^2 + dy^2 + \left(1 - \frac{2Mr}{z^2+a^2}\right)^{-1} dz^2
\end{aligned} \tag{19}
$$

The form of the metric on the symmetry axis given by the last equation is the one that allows for easy comparison with the Schwarschild metric to which it reduces when $a = 0$.

Of course, when $M^2 \geqslant a^2$, the function $\Delta(r)$ vanishes at $r = r_{\pm} := M \pm (M^2 - a^2)^{1/2}$. In this case the Kerr metric represents a rotating black hole and its Killing vector $\xi^a := \delta_0{}^a$ is not timelike in the region $r_- < r < r_+$ between the inner and outer horizon. As a result, the tetrad (15) can be interpreted as a static frame nonrotating with respect to the asymptotic regions of the Kerr spacetime only for $|z| > r_+$.

To find the 4-acceleration of the above frame as well as its rotation with respect to a triad of inertial gyroscopes carried by the static frame itself, we can use the results of Appendix A. Denoting covariant differentiation by a semicolon, we have

$$
\begin{aligned}
a^a &:= e_{\hat{0}}{}^a{}_{;b}\, e_{\hat{0}}{}^b = \frac{Mz(z^2-a^2)\, H}{r(z^2+a^2)^2}\, e_{\hat{3}}{}^a \\
e_{\hat{1}}{}^a{}_{;b}\, e_{\hat{0}}{}^b &= -\frac{2MarH}{(z^2+a^2)^2}\, e_{\hat{2}}{}^a \\
e_{\hat{2}}{}^a{}_{;b}\, e_{\hat{0}}{}^b &= \frac{2MarH}{(z^2+a^2)^2}\, e_{\hat{1}}{}^a \\
e_{\hat{3}}{}^a{}_{;b}\, e_{\hat{0}}{}^b &= \frac{Mz(z^2-a^2)\, H}{r(z^2+a^2)^2}\, e_{\hat{0}}{}^a
\end{aligned} \tag{20}
$$

where

$$
H := [(z^2+a^2)/\Delta]^{1/2} \tag{21}
$$

These results, first obtained by Urani and Carlson [8] using the "isometry method" of Urani and Kemp [9], bring out clearly the effect of the gravitational source's angular momentum $S := Ma$ on the motion of particles and frames. Thus, the first of equations (20) shows that, along the symmetry axis of the Kerr spacetime, the gravitational field is attractive as long as $|z| > |a|$. But at $|z| = |a|$ it reverses its direction to become repulsive in the central portion, $-|a| < z < |a|$, of the axis. At the antipodal points $z = \pm|a|$, a particle is falling freely by not falling at all.

The last three of (20), on the other hand, express the well-known "dragging of frames" effect associated with a rotating source of gravity. Specifically, they show that the static frame (15) is rotating with an angular velocity $\Omega_S{}^a$, where

$$\Omega_S{}^a = -\frac{2SrH}{(z^2+a^2)^2} e_{\hat{3}}{}^a \tag{22}$$

relative to a frame of inertial guidance gyroscopes located at the same point of the symmetry axis of the Kerr spacetime. Since the factor H represents the ratio $(dt/d\tau_s)$ of the asymptotic to the static frame proper time, we can equivalently say that the gyroscopes are rotating with an angular velocity $\Omega_G{}^a$, where

$$\Omega_G{}^a = \frac{2Sr}{(z^2+a^2)^2} e_{\hat{3}}{}^a \tag{23}$$

relative to the fixed stars.

Let us also consider a particle moving in the Kerr field along an orbit characterized by $r = \text{const.}$, $\theta = \text{const.}$, and

$$d\varphi/dt = w(r, \theta) := -g_{t\varphi}/g_{\varphi\varphi} = 2Mar/A \tag{24}$$

in the BL coordinates.

An orthonormal frame comoving with a particle which has its axes $e_{\hat{r}}{}^a$, $e_{\hat{\theta}}{}^a$, and $e_{\hat{\varphi}}{}^a$ pointing in the direction of the r, θ, and φ coordinates, respectively, is called a locally nonrotating frame (LNRF). Such frames were first introduced by Bardeen [10, 11] and they proved to be very useful in the analysis of gravitational effects in the Kerr spacetime. A LNRF is not inertial, in general. Indeed, using the pertinent results of references [11] and [7, p. 290], one can show that the 4-acceleration $a^a{}_{\text{LNRF}}$ and angular velocity relative to comoving gyroscopes $\Omega^a{}_{\text{LNRF}}$ of a LNRF are given by

$$a^a{}_{\text{LNRF}} = (\Delta/\Sigma)^{1/2}\left[\frac{r}{\Sigma} + \frac{r-M}{\Delta} - \frac{2r(r^2+a^2) - a^2(r-M)\sin^2\theta}{A}\right] e_{\hat{r}}{}^a \tag{25}$$

and

$$\Omega^a{}_{\text{LNRF}} = \frac{2\Delta^{1/2} M a^3 r \sin^2\theta \cos\theta}{A\Sigma^{3/2}} e_{\hat{r}}{}^a$$
$$+ \frac{Ma \sin\theta}{A\Sigma^{3/2}} [(r^2 + a^2)(3r^2 - a^2) + (r^2 - a^2)\, a^2 \sin^2\theta]\, e_{\hat{\theta}}{}^a \tag{26}$$

respectively. Therefore

$$\lim_{\sin\theta \to 0} a^a{}_{\text{LNRF}} = \frac{M(r^2 - a^2)\, H}{(r^2 + a^2)^2} e_{\hat{r}}{}^a \tag{27}$$

and

$$\lim_{\sin\theta \to 0} \Omega^a{}_{\text{LNRF}} = 0 \tag{28}$$

On the other hand, it follows from (24) that

$$\lim_{\sin\theta \to 0} \frac{d\varphi}{dt} = \frac{2Sr}{(z^2 + a^2)^2} \tag{29}$$

As expected, (27) agrees with the first equation of (20), while (28), (29) along with (22) show that on the symmetry axis a LNRF is a frame of gyroscopes rotating at a rate of $2Sr/(z^2 + a^2)^2$ about the symmetry axis itself.

Let us now withdraw the support that keeps the given particle at a fixed point of the symmetry axis of the Kerr spacetime. The particle will then follow a timelike geodesic of the submanifold $x = y = 0$. Therefore, the particle's 4-velocity vector u^a satisfies the condition $u^a u_a = -1$, where $u^a = \mathring{x}^a := dx^a/d\tau$ with τ denoting proper time. Moreover, due to the stationary character of the given spacetime, the particle's energy per unit rest mass at infinity $E = -u_0 = \text{const}$. Then the particle's orbit is given by the first integrals [2]

$$\mathring{t} = H^2 E \tag{30}$$

and

$$\mathring{z} = \pm [E^2 - V^2(r)]^{1/2} \tag{31}$$

where

$$V^2(r) := 1 - 2Mr/(r^2 + a^2) \tag{32}$$

Equation (32) shows that $V^2(r)$ plays the role of a squared potential for a particle falling freely along the axis and allows for easy visualization of the orbit's qualitative features [2]. Equation (30), on the other hand, shows that one has to distinguish between the cases $a^2 > M^2$ and $a^2 \leqslant M^2$. In the former, the geodesic corresponding to the freely falling particle remains in a bounded region of the t–z plane. Depending on whether $\Gamma := 1 - E^2$ is negative, zero, or positive, the particle's orbit extends over the entire z axis. It is reflected at $z = 0$ having started at $z = \pm\infty$ or oscillates within a finite region of the positive or negative part of the axis, respectively. Thus, for Γ, $z > 0$ we have $r_{\langle} \leqslant z(\tau) \leqslant r_{\rangle}$, where $r_{\rangle,\langle} = (M/\Gamma)\{1 \pm [1 - (\Gamma a/M)^2]^{1/2}\}$. The closed interval $[r_{\langle}, r_{\rangle}]$ includes the point $r = |a|$ where $V^2(r)$ has the minimum value of $1 - M/|a|$ and degenerates to this point when $E^2 = 1 - M/|a|$. This corresponds to the exceptional case of the particle which is freely falling by not falling at all discussed earlier in this section.

In the case of a rotating black hole $M^2 \geqslant a^2$, the (30) shows that the coordinate time of the freely falling particle becomes unbounded as the particle approaches the horizon. This, of course, is a coordinate singularity, but its influence on the orbit is quite significant. Consider, for example, a particle oscillating between $r_{\langle}$ and $r_{\rangle}$ as above. When $M^2 > a^2$, the point $r_{\rangle}$ lies in region $I_{\rangle}$, where $r > r_+$, while $r_{\langle}$ lies in region $I_{\langle}$, where $r < r_-$. Thus, each time the particle moves from $I_{\rangle}$ to $I_{\langle}$ or vice-versa, it passes through region $I_{|a|}$, which lies between the inner and outer horizons. Carter's analytic extension, however, is such that at each crossing of a horizon the particle moves into a new copy of the region into which it enters. As a consequence, the particle oscillating between $r_{\langle}$ and $r_{\rangle}$ moves along a sequence of spacetime regions of the form $I_{\langle}$, $I_{|a|}$, $I_{\rangle}$, $I'_{|a|}$, $I'_{\langle}$, never returning to its starting point. Region $I_{|a|}$ reduces to the point $r = |a|$ when $M^2 = a^2$, but the existence of only one horizon instead of two does not change the effect described above. Let us note finally that region $I_{\langle}$ includes the points where $r < 0$. It then follows from (32) that $V^2(r)$ has a maximum of $1 + M/|a|$ at the point $r = -|a|$ of $I_{\langle}$. As a result, a particle which at some instant was at a point with $r > -|a|$ can move over to the $r < -|a|$ side of region $I_{\langle}$ only if its energy squared exceeds $(1 + M/|a|)\, m^2$.

Using (15), (16), (30)–(32) we can write the 4-velocity vector of a freely falling particle in the form

$$u^a = P e_{\hat{0}}{}^a + Q e_{\hat{3}}{}^a \tag{33}$$

where

$$\begin{aligned} P &:= HE \\ Q &:= \varepsilon(\mathring{z})(H^2E^2 - 1)^{1/2} \end{aligned} \tag{34}$$

and $\varepsilon(\mathring{z}) = -1, 0, 1$, depending on whether $\mathring{z}$ is less, equal, or greater than zero, respectively. Since

$$P^2 - Q^2 = 1 \tag{35}$$

the vector $\lambda_{(3)}{}^a$ defined by

$$\lambda_{(3)}{}^a := Qe_{\hat{0}}{}^a + Pe_{\hat{3}}{}^a \tag{36}$$

forms with $e_{\hat{1}}{}^a$ and $e_{\hat{2}}{}^a$ an orthonormal frame comoving with the freely falling particle.

From Appendix A we find that

$$e_{\hat{1}}{}^a{}_{;b}e_{\hat{3}}{}^b := \frac{2MazH}{(z^2+a^2)^2}e_{\hat{2}}{}^a \tag{37}$$

Combining this equation with (20) we obtain

$$\mathring{e}_{\hat{1}}{}^a := e_{\hat{1}}{}^a{}_{;b}u^b = -\Omega_C e_{\hat{2}}{}^a \tag{38}$$

where

$$\Omega_C = \frac{2MarH}{(z^2+a^2)^2}\left[P-(z/r)\,Q\right] \tag{39}$$

In a similar fashion we find that

$$\mathring{e}_{\hat{2}}{}^a := \Omega_C e_{\hat{1}}{}^a \tag{40}$$

Equations (38) and (20) show that the comoving frame $\{e_{\hat{1}}{}^a, e_{\hat{2}}{}^a, \lambda_{(3)}{}^a\}$ rotates about $\lambda_{(3)}{}^a$ with angular velocity equal to Ω_c relative to a set of gyroscopes carried by the freely falling particle. Equivalently, the frame $\{\lambda_{(i)}{}^a\}$, where

$$\begin{aligned}\lambda_{(1)}{}^a &:= \cos\Psi(\tau)\,e_{\hat{1}}{}^a + \sin\Psi(\tau)\,e_{\hat{2}}{}^a\\ \lambda_{(2)}{}^a &:= -\sin\Psi(\tau)\,e_{\hat{1}}{}^a + \cos\Psi(\tau)\,e_{\hat{2}}{}^a\end{aligned} \tag{41}$$

$\lambda_{(3)}{}^a$ is defined by (36) and

$$\mathring{\Psi} = \Omega_C \tag{42}$$

is an inertial frame associated with a particle of energy at infinity $\bar{E} = mE$ falling freely along the symmetry axis of the Kerr spacetime.

4. THE TIDAL TENSOR

In the freely falling nonrotating frame $\{\lambda_{(i)}{}^a\}$ constructed in the previous section the gravitational field vanishes (principle of equivalence). Therefore, a physicist riding such a frame has to measure the relative acceleration of free particles in order to determine whether the spacetime is actually curved or not. This acceleration is determined by the tidal tensor [12] $C_{(i)(j)}$ which is defined by

$$C_{(i)(j)} = R_{abcd} u^a \lambda_{(i)}{}^b u^c \lambda_{(i)}{}^d \tag{43}$$

where R_{abcd} is the curvature tensor of the spacetime in question.

In the case of the Kerr spacetime, it is of particular interest to see how the magnetic components g_{0i} of the metric, which are responsible for the frame-dragging effects, show their presence within an inertial frame. Since the influence of these terms is also incorporated in the tidal tensor, we turn to calculating its components.

Using the results of Appendix B we find that the nonvanishing components of the curvature tensor with respect to the static frame (15) are given by

$$\begin{aligned} R_{\hat{0}\hat{1}\hat{0}\hat{1}} &= R_{\hat{0}\hat{2}\hat{0}\hat{2}} = -R_{\hat{0}\hat{3}\hat{0}\hat{3}}/2 = R_{\hat{1}\hat{2}\hat{1}\hat{2}}/2 = -R_{\hat{2}\hat{3}\hat{2}\hat{3}} = -R_{\hat{3}\hat{1}\hat{3}\hat{1}} \\ &= \frac{Mr(z^2-3a^2)}{(z^2+a^2)^3} =: I_1 \\ R_{\hat{0}\hat{1}\hat{2}\hat{3}} &= R_{\hat{0}\hat{2}\hat{3}\hat{1}} = -R_{\hat{0}\hat{3}\hat{1}\hat{2}}/2 = \frac{Maz(a^2-3z^2)}{r(z^2+a^2)^3} =: I_2 \end{aligned} \tag{44}$$

and the symmetries of the tensor R_{abcd}. From these expressions and (33)–(36) we find that

$$C_{(i)(j)} = \mathrm{diag}(I_1, I_1, -2I_1) \tag{45}$$

In order to obtain a physical intuition of this result, let us compare it with the case of an inertial frame falling along a radial direction of the Schwarzshild spacetime [13]. The latter is obtained by simply setting $a=0$ in the equations obtained so far. Thus, if we denote the corresponding tidal tensor by $C^s{}_{(i)(j)}$, we will have

$$C^s{}_{(i)(j)} = \mathrm{diag}\ (I_1{}^s, I_1{}^s, -2I_1{}^s) \tag{46}$$

where

$$I_1{}^s := M/r^3 \tag{47}$$

It is well-known that (46) implies the following. The radial direction along which the frame's origin falls can be identified with the z axis. Then, free particles which are found on the z axis and near the origin will be repelled from the latter while those that lie in the x–y plane of the frame will be attracted toward it. As a result, a set of test particles which at a given instant had vanishing velocity relative to the inertial frame and formed a spherical shell centered at the frame's origin will, at a later instant, form a surface of revolution elongated along the z axis.

Turning to (45), we see that the behavior of test particles near the origin of the inertial frame which falls along the symmetry axis of the Kerr spacetime is essentially the same with the one described above corresponding to the case of radial free fall in the spherically symmetric Schwarschild field. With a big difference. As a result of the nonvanishing of a, I_1 changes sign at $|z| = 3^{1/2}\,|a|$. This implies that, as the inertial frame moves from the region where $|z| > 3^{1/2}\,|a|$ into the region where $|z| < 3^{1/2}\,|a|$, a reversal of the tidal forces direction takes place. Thus, test particles lying on the frame's z axis are now attracted toward the origin while those on the x–y plane are repelled from it. It should be noted that, provided $a^2 > (3M^2/4)$, the tidal forces turning point $|z| = 3^{1/2}\,|a|$ lies outside the event horizon which crosses the z axis at $|z| = r_+$ in the case of a rotating black hole. Thus, when the above condition on the angular momentum parameter is satisfied, a rocket ship can enter the region where the tidal forces operate in the reverse of the usual fashion without running the risk of being swallowed by the black hole.

APPENDIX A

Faridi [14] has given the connection coefficients $\Gamma^{\alpha}{}_{bc}$ of the Kerr metric in the KS system of coordinates in terms of the null vector l^a. In this appendix we present an outline of Faradi's method for calculating $\Gamma^{\alpha}{}_{bc}$ and give explicit expressions for $\Gamma_{abc} := g_{ad}\Gamma^{d}{}_{bc}$ valid on the symmetry axis of the Kerr spacetime.

From (9) and the fact that l^a is a null vector with respect to the Minkowski metric it follows that

$$\delta_{ij} l^i l^j = 1 \tag{A1}$$

where $i, j = 1, 2, 3$. This implies that

$$l^i l_{i,j} = 0 \tag{A2}$$

where $(\ )_{,i} := \partial(\ )/\partial x^i$. Therefore

$$l_{i,j} = l_{(i,j)} + l_{[i,j]} = \alpha(\delta_{ij} - l_i l_j) + \beta \varepsilon_{ijk} l^k \tag{A3}$$

where round and square brackets denote the symmetric and antisymmetric part, respectively, and ε_{ijk} is the totally antisymmetric symbol with $\varepsilon_{123} = 1$.

The functions α and β can be determined easily from $l_3 = z/r$ and the equation

$$r,_i = L^2[x_i + z(a/r)^2\, \delta^3{}_i] \tag{A4}$$

which follows from (4). The result reads

$$l_{i,j} = L^2\left[(\delta_{ij} - l_i l_j) - \frac{az}{r^2}\varepsilon_{ijk} l^k\right] \tag{A5}$$

From (11), on the other hand, one obtains

$$L^2,_i = L^4\left[\left(\frac{3a^2z^2}{r^4} - 1\right) r,_i - 2\frac{a^2z}{r^3}\delta_i^3\right] \tag{A6}$$

The form (10) of the Kerr metric shows that equations (A3)–(A6) suffice for the calculation of Γ_{abc}. Taking into account the fact that on the axis, where $|z| = r$

$$L^2 \stackrel{*}{=} r(z^2 + a^2)^{-1}$$

$$l_i \stackrel{*}{=} r,_i = (z/r)\,\delta_i^3$$

$$L^2,_i \stackrel{*}{=} (z/r)(a^2 - z^2)(z^2 + a^2)^{-2}\,\delta_i^3$$

and

$$l_{i,j} \stackrel{*}{=} [r(\delta_{ij} - \delta_i^3\delta_j^3) - a\varepsilon_{ij3}](z^2 + a^2)^{-1} \tag{A7}$$

one obtains

$$\Gamma_{000} = ML^2,_0 = 0$$

$$\Gamma_{00i} = -\Gamma_{i00} = ML^2,_i \stackrel{*}{=} M(z/r)(a^2 - z^2)(z^2 + a^2)^{-2}\,\delta_i^3$$

$$\Gamma_{0ij} = -2M\{L^2 l_{(i,j)} + l_{(i}L^2,_{j)}\}$$

$$\stackrel{*}{=} -2Mz^2(z^2 + a^2)^{-2}\,\delta_{ij} + 2M(2z^2 - a^2)(z^2 + a^2)^{-2}\,\delta_i^3\delta_j^3$$

$$\Gamma_{ij0} = -2M(L^2 l_{[i,j]} + l_{[i}L^2,_{j]}) \stackrel{*}{=} 2Mar\,(z^2 + a^2)^{-2}\,\varepsilon_{ij3}$$

$$\Gamma_{ijk} = M\{l_i l_j L^2,_k + l_k l_i L^2,_j - l_j l_k L^2,_i + 2L^2(l_i l_{(j,k)} + l_j l_{[i,k]} + l_k l_{[i,j]})\}$$

$$= M(z/r)(z^2 + a^2)^{-2}\,[2z^2\,\delta_i^3\delta_{jk}$$

$$+ (a^2 - 3z^2)\,\delta_i^3\delta_j^3\delta_k^3 - 2ar(\delta_j^3\varepsilon_{ik3} + \delta_k^3\varepsilon_{ij3})] \tag{A8}$$

APPENDIX B

The curvature tensor components R_{abcd} can be calculated using the formula [15]

$$R_{abcd} = \tfrac{1}{2}(g_{ad,bc} + g_{bc,ad} - g_{ac,bd} - g_{bd,ac}) + \Gamma_{ebc}\Gamma^{e}_{ad} - \Gamma_{ebd}\Gamma^{e}_{ac} \tag{B1}$$

Beyond the quantities given in the previous appendix, one needs the second partial derivatives of L^2 and l_i. A calculation similar to the one that led to (A8) gives the following results

$$L^2{}_{,ij} \overset{*}{=} r(3a^2 - z^2)(z^2 + a^2)^{-3}\,(\delta_{ij} - 3\delta_i{}^3\delta_j{}^3)$$

$$\begin{aligned} l_{i,jk} \overset{*}{=} {} & (z/r)(z^2+a^2)^{-2}\,[(a^2+z^2)\,\delta_{ij}\delta_k{}^3(a^2+z^2)\,\delta_{jk}\delta_i{}^3 + (a^2+z^2)\,\delta_{jk}\delta_i{}^3 \\ & + (3z^2 - a^2)\,\delta_i{}^3\delta_j{}^3\delta_k{}^3 + ar(3\varepsilon_{ij3}\delta_k{}^3 + \varepsilon_{ik3}\delta_i{}^3 + \varepsilon_{ik3}\delta_j{}^3 - \varepsilon_{ijk})] \end{aligned} \tag{B2}$$

Using the above equations, one finds that the nonvanishing components are those that can be obtained from the list that follows and the symmetries of the curvature tensor.

$$\begin{aligned} & R_{0101} \overset{*}{=} R_{0202} \overset{*}{=} [\Delta/(z^2+a^2)]\,I_1, \qquad R_{0303} \overset{*}{=} -R_{1212} \overset{*}{=} -2I_1 \\ & R_{0123} \overset{*}{=} R_{0231} \overset{*}{=} -R_{0312}/2 = I_2, \qquad R_{0131} \overset{*}{=} R_{0223} \overset{*}{=} [2Mz/(z^2+a^2)]\,I_1 \\ & R_{1313} \overset{*}{=} R_{2323} \overset{*}{=} -\left(1 + \frac{2Mz}{z^2+a^2}\right) I_1 \end{aligned} \tag{B3}$$

where

$$I_1 := \frac{Mr(z^2 - 3a^2)}{(z^2+a^2)^3}, \qquad I_2 := \frac{Maz(a^2 - 3z^2)}{r(z^2+a^2)^3} \tag{B4}$$

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